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The Bremsstrahlung Equation for the Spin Motion in Electromagnetic Field

Miroslav Pardy

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Abstract The influence of the bremsstrahlung on the spin motion is expressed by the equation which is the analogue and generalization of the Bargmann-Michel-Telegdi equation. The new constant is involved in this equation. This constant can be immediately determined by the experimental measurement of the spin motion, or it follows from the classical limit of quantum electrodynamics with radiative corrections.

Keywords Bargmann-Michel-Telegdi equation · Synchrotron radiation · Spin light

1 Introduction

The synchrotron radiation evidently influences the motion of the electron in accelerators. The corresponding equation which describes the classical motion is so called the Lorentz-Dirac equation, which differs from the so called Lorentz equation only by the additional term which describes the radiative corrections. The equation with the radiative term is as follows [8]:

$$mc\frac{du_\mu}{ds} = \frac{e}{c}F_{\mu\nu}u^\nu + g_\mu, \tag{1}$$

where u_μ is the four-velocity and the radiative term was derived by Landau et al. in the form [8]:

$$g_\mu = \frac{2e^3}{3mc^3}\frac{\partial F_{\mu\nu}}{\partial x^\alpha}u^\nu u^\alpha - \frac{2e^4}{3m^2c^5}F_{\mu\alpha}F^{\beta\alpha}u_\beta + \frac{2e^4}{3m^2c^5}(F_{\alpha\beta}u^\beta)(F^{\alpha\gamma}u_\gamma)u_\mu. \tag{2}$$

M. Pardy (✉)
Department of Physical Electronics and Laboratory of Plasma Physics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
e-mail: pamir@physics.muni.cz

It is possible to show that the space components of the 4-vector force g_μ is of the form [8]

$$\mathbf{f} = \frac{2e^3}{3mc^3}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\left\{\left(\frac{\partial}{\partial t}+(\mathbf{v}\nabla)\right)\mathbf{E}+\frac{1}{c}\left[\mathbf{v}\left(\frac{\partial}{\partial t}+(\mathbf{v}\nabla)\right)\mathbf{H}\right]\right\}$$
$$+\frac{2e^4}{3m^2c^3}\left\{\mathbf{E}\times\mathbf{H}+\frac{1}{c}\mathbf{H}\times(\mathbf{H}\times\mathbf{v})+\frac{1}{c}\mathbf{E}(\mathbf{vE})\right\}$$
$$-\frac{2e^4}{3m^2c^5(1-\frac{v^2}{c^2})}\mathbf{v}\left\{\left(\mathbf{E}+\frac{1}{c}(\mathbf{v}\times\mathbf{H})\right)^2-\frac{1}{c^2}(\mathbf{Ev})^2\right\}. \quad (3)$$

Bargmann, Michel and Telegdi [4] derived so called BMT equation for motion of spin in the electromagnetic field, in the form

$$\frac{da_\mu}{ds} = \alpha F_{\mu\nu}a^\nu - \beta u_\mu F^{\nu\lambda}u_\nu a_\lambda, \quad (4)$$

where a_μ is so called axial vector describing the classical spin and constants α and β were determined after the comparison of the postulated equations with the non-relativistic quantum mechanical limit. The result of such comparison is the final form of so called BMT equations:

$$\frac{da_\mu}{ds} = 2\mu F_{\mu\nu}a^\nu - 2\mu' u_\mu F^{\nu\lambda}u_\nu a_\lambda, \quad (5)$$

where μ is magnetic moment of electron following directly from the Dirac equation and μ' is anomalous magnetic moment of electron which can be calculated as the radiative correction to the interaction of electron with electromagnetic field and follows from quantum electrodynamics. The BMT equation has more earlier origin. The first attempt to describe the spin motion in electromagnetic field using the special theory of relativity was performed by Thomas [14]. However, the basic ideas on the spin motion was established by Frenkel [6, 7]. After appearing the Frenkel basic article, many authors published the articles concerning the spin motion [12, 15]. The mechanical model of spin was constructed by Uhlenbeck and Goudsmith [16], or, in the very sophisticated form by Ohanian [9] and other authors. However, we know at present time that spin of electron is its physical attribute which follows only from the Dirac equation. Also the Schrödinger Zitterbewegung of the Dirac electron as a point-like particle follows from the Dirac equation.

It was shown by Rafanelli and Schiller [11], [10] that the BMT equation can be derived from the classical limit, i.e. from the WKB solution of the Dirac equation with the anomalous magnetic moment. Equation (5) is also the basic equation of the non-dissipative spintronics.

2 Equation of Motion for the Spin-Vector

If we introduce the average value of the vector of spin in the rest system by the quantity $\boldsymbol{\zeta}$, then the 4-pseudovector a^μ is of the from $a^\mu = (0, \boldsymbol{\zeta})$. The momentum four-vector of a particle is $p^\mu = (m, 0)$ in the rest system of a particle. Then the equation

$$a^\mu p_\mu = 0 \quad (6)$$

is valid not only in the rest system of a particle but in the arbitrary system as a consequence of the relativistic invariance. The following general formula is also valid in the arbitrary system

$$a^\mu a_\mu = -\boldsymbol{\zeta}^2. \tag{7}$$

The components of the axial 4-vector a^μ in the reference system where particle is moving with the velocity $\mathbf{v} = \mathbf{p}/\varepsilon$ can be obtained by application of the Lorentz transformation to the rest system and they are as follows [4]:

$$a^0 = \frac{|\mathbf{p}|}{m}\boldsymbol{\zeta}_\parallel, \qquad \mathbf{a}_\perp = \boldsymbol{\zeta}_\perp, \qquad a_\parallel = \frac{\varepsilon}{m}\boldsymbol{\zeta}_\parallel, \tag{8}$$

where suffices $\parallel, \perp$ denote the components of $\mathbf{a}$, $\boldsymbol{\zeta}$ parallel and perpendicular to the direction $\mathbf{p}$. The formulas for the components can be also rewritten in the more compact form as follows [4]:

$$\mathbf{a} = \boldsymbol{\zeta} + \frac{\mathbf{p}(\boldsymbol{\zeta}\mathbf{p})}{m(\varepsilon+m)}, \qquad a^0 = \frac{\mathbf{ap}}{\varepsilon} = \frac{\boldsymbol{\zeta}\mathbf{p}}{m}, \qquad \mathbf{a}^2 = \boldsymbol{\zeta}^2 + \frac{(\mathbf{p}\boldsymbol{\zeta})^2}{m^2}. \tag{9}$$

The equation for the change of polarization can be obtained immediately from the BMT equation in the following form [4]:

$$\begin{aligned}\frac{d\mathbf{a}}{dt} &= \frac{2\mu m}{\varepsilon}\mathbf{a}\times\mathbf{H} + \frac{2\mu m}{\varepsilon}(\mathbf{av})\mathbf{E} - \frac{2\mu'\varepsilon}{m}\mathbf{v}(\mathbf{aE}) \\ &\quad + \frac{2\mu'\varepsilon}{m}\mathbf{v}(\mathbf{v}(\mathbf{a}\times\mathbf{H})) + \frac{2\mu'\varepsilon}{m}\mathbf{v}(\mathbf{av})(\mathbf{vE}),\end{aligned} \tag{10}$$

where we used the relativistic relations $c = 1$, $ds = dt\sqrt{1-v^2}$, $\varepsilon = m\sqrt{1-v^2}$ and the following components of the electromagnetic field [8]:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} \stackrel{d}{=} (\mathbf{E},\mathbf{H}); \qquad F_{\mu\nu} = (-\mathbf{E},\mathbf{H}). \tag{11}$$

Inserting equation $\mathbf{a}$ from (9) into (10) and using equations

$$\mathbf{p} = \varepsilon\mathbf{v}, \qquad \varepsilon^2 = \mathbf{p}^2 + m^2, \qquad \frac{d\mathbf{p}}{dt} = e\mathbf{E} + e(\mathbf{v}\times\mathbf{H}), \qquad \frac{d\varepsilon}{dt} = e(\mathbf{vE}), \tag{12}$$

we get after long but simple mathematical operations the following equation for the polarization $\boldsymbol{\zeta}$

$$\begin{aligned}\frac{d\boldsymbol{\zeta}}{dt} &= \frac{2\mu m + 2\mu'(\varepsilon - m)}{\varepsilon}\boldsymbol{\zeta}\times\mathbf{H} \\ &\quad + \frac{2\mu'\varepsilon}{\varepsilon+m}(\mathbf{vH})(\mathbf{v}\times\boldsymbol{\zeta}) + \frac{2\mu m + 2\mu'\varepsilon}{\varepsilon+m}\boldsymbol{\zeta}\times(\mathbf{E}\times\mathbf{v}).\end{aligned} \tag{13}$$

The special interest is concerned not only in the change of the absolute quantity of the polarization, but in the change with regard to the direction of motion represented by the unit vector $\mathbf{n} = \mathbf{v}/v$. We write the polarization in the form:

$$\boldsymbol{\zeta} = \mathbf{n}\zeta_\parallel + \boldsymbol{\zeta}_\perp. \tag{14}$$

Then using (12), (13) and (14), we get the following equation for the parallel component of the polarization [4]:

$$\frac{d\zeta_{\parallel}}{dt} = 2\mu'(\boldsymbol{\zeta}_{\perp}(\mathbf{H}\times\mathbf{n})) + \frac{2}{v}\left(\frac{\mu m^2}{\varepsilon^2} - \mu'\right)(\boldsymbol{\zeta}_{\perp}\mathbf{E}). \tag{15}$$

3 Spin Motion Equation with the Bremsstrahlung Reaction

It is meaningful to consider the BMT equation with the radiative corrections to express the influence of the synchrotron radiation on the motion of spin. To our knowledge such equation, the generalized BMT equation, was not published and we here present the conjecture of the form of such equation. The equation of the spin motion under the influence of the synchrotron radiation is suggested as an analogue to the BMT construction:

$$\frac{da_\mu}{ds} = 2\mu F_{\mu\nu}a^\nu - 2\mu' u_\mu F^{\nu\lambda}u_\nu a_\lambda + \Lambda f_\mu(axial), \tag{16}$$

where the term $f_\mu(axial)$ is generated as the "axialization" of the force elaborated from the radiation term g_μ. The axialization is the operation which was used by Bargmann, Michel and Telegdi and it consists in the construction of the axial vector from the four-vector force. We see from the right side of the BMT equation how to construct such axial equation. Or, the additional axial 4-vector constructed from the bremsstrahlung force is as following:

$$f_\mu(axial) = \Lambda u_\mu(g^\alpha a_\alpha) = \Lambda u_\mu[g_0 a_0 - \mathbf{g}\cdot\mathbf{a}]. \tag{17}$$

So, the generalized BMT equation which involves also the influence of synchrotron radiation on spin motion is as follows:

$$\begin{aligned}\frac{da_\mu}{ds} &= 2\mu F_{\mu\nu}a^\nu - 2\mu' u_\mu F^{\nu\lambda}u_\nu a_\lambda \\ &\quad + \Lambda u_\mu\left\{\frac{2e^3}{3mc^3}\frac{\partial F_{\lambda\nu}}{\partial x^\alpha}u^\nu u^\alpha\right. \\ &\quad \left. - \frac{2e^4}{3m^2c^5}F_{\lambda\alpha}F^{\beta\alpha}u_\beta + \frac{2e^4}{3m^2c^5}(F_{\alpha\beta}u^\beta)(F^{\alpha\gamma}u_\gamma)u_\lambda\right\}a^\lambda.\end{aligned} \tag{18}$$

Using (17), we can write (18) in the form

$$\frac{da_\mu}{ds} = 2\mu F_{\mu\nu}a^\nu - 2\mu' u_\mu F^{\nu\lambda}u_\nu a_\lambda + \Lambda u_\mu[g_0 a_0 - \mathbf{g}\cdot\mathbf{a}]. \tag{19}$$

The constant Λ is new physical constant, which cannot be determined from the classical theory of the spin motion. This constant can be determined immediately from the spin motion observed experimentally. However, this constant follows logically from the classical limit of quantum electrodynamics (QED) involving radiative corrections. The solution of this problem in the framework of the WKB limit of the Dirac equation with radiation term was not still published. On the other hand, Bayer [2], Bayer et al. [3] derived, by the different method, the equation of the spin motion in electromagnetic field where the influence of radiative reaction on the spin motion is involved. This equation was used later for the determination of the polarization of electrons in the bent crystals [1].

While the 3-vector components of the radiative force are involved in (3) the zero component must be determined by the extra way. We have:

$$g_0 = P_1 + P_2 + P_3, \tag{20}$$

where the terms of (20) follow from (2) in the form ($c = 1$):

$$P_1 = \left(\frac{2e^3}{3m}\right)\frac{\partial F_{0\nu}}{\partial x^\alpha}u^\nu u^\alpha, \tag{21}$$

$$P_2 = \left(-\frac{2e^4}{3m^2}\right)F_{0\alpha}F^{\beta\alpha}u_\beta, \tag{22}$$

$$P_3 = \left(\frac{2e^4}{3m^2}\right)(F_{\alpha\beta}u^\beta)(F^{\alpha\gamma}u_\gamma)u_0 \tag{23}$$

with

$$u = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\mathbf{v}}{\sqrt{1-v^2}}\right). \tag{24}$$

After some algebraic operations, we write the set of quantities P_1, P_2, P_3 as follows:

$$P_1 = \left(\frac{2e^3}{3m}\right)\frac{1}{1-v^2} \times \{(\partial_t \mathbf{E})\cdot\mathbf{v} + (\partial_x \mathbf{E})\cdot\mathbf{v}v_x + (\partial_y \mathbf{E})\cdot\mathbf{v}v_y + (\partial_z \mathbf{E})\cdot\mathbf{v}v_z\}, \tag{25}$$

$$P_2 = \left(\frac{2e^4}{3m^2}\right)\frac{1}{\sqrt{1-v^2}}\{E^2 - (\mathbf{H}\times\mathbf{E})\cdot\mathbf{v})\} \tag{26}$$

$$P_3 = \left(-\frac{2e^4}{3m^2}\right)\frac{1}{(1-v^2)^{3/2}}\{(\mathbf{E} + (\mathbf{v}\times\mathbf{H}))^2 - (\mathbf{E}\cdot\mathbf{v})^2\}. \tag{27}$$

The relation of this equation to the (dissipative) spintronics cannot be a priori excluded. Such equation will have fundamental meaning for the work of LHC where the synchrotron radiation influences the spin motion of protons in LHC.

4 The General Solution of the Spin Precession Equation

Equation (13) involving the 3-vector of the radiation term (17) can be in general written in the following form:

$$\frac{d}{dt}\zeta_k = \sum_{l=1}^{3} a_{kl}\zeta_l + \Lambda\sum_{l=1}^{3} b_{kl}\zeta_l, \tag{28}$$

where the coefficients a_{kl} are the corresponding coefficient in (13) and b_{kl} are the corresponding coefficient in (17).

It follows from the theory of the differential equations that the solution of the system (28) is in general of the following form:

$$\zeta_k(t) = \alpha_k e^{i\Omega t}, \tag{29}$$

where α_k and Ω are some constants. The time derivative of (29) is now

$$\frac{d\zeta_k}{dt} = \alpha_k(i\Omega)e^{i\Omega t}. \tag{30}$$

After insertion of (29) and (30) into (28), we get the following system after some elementary modification:

$$(a_{11} + \Lambda b_{11} - i\Omega)\alpha_1 + (a_{12} + \Lambda b_{12})\alpha_2 + (a_{13} + \Lambda b_{13})\alpha_3 = 0 \tag{31a}$$

$$(a_{21} + \Lambda b_{21})\alpha_1 + (a_{22} + \Lambda b_{22} - i\Omega)\alpha_2 + (a_{23} + \Lambda b_{23})\alpha_3 = 0 \tag{31b}$$

$$(a_{31} + \Lambda b_{31})\alpha_1 + (a_{32} + \Lambda b_{32})\alpha_2 + (a_{33} + \Lambda b_{33} - i\Omega)\alpha_3 = 0. \tag{31c}$$

The nontrivial solution of the system (31) for the determination of α_i is possible if and only if the determinate of the system is zero, or,

$$\det(A + \Lambda B - i\Omega E) = 0, \tag{32}$$

where A, B, E are matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{33}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{34}$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

Equation (32) is the equation for the determination of the three complex frequencies $\Omega_k = \Omega_1, \Omega_2, \Omega_3$. To the every frequency corresponds the solution

$$\zeta_k(t) = \beta_k e^{i\Omega_k t} \tag{36}$$

and it means that the solution of the system (31) is given as the linear combination of the particular solutions. Or,

$$\zeta_k = \sum_{l=1}^{3} \beta_{kl} e^{i\Omega_l(\Lambda)t}, \tag{37}$$

where β_{kl} are some coefficients which can be determined by insertion of (37) in (31).

However, Ω-s are the complex quantities depending on the small parameter Λ. So we can write:

$$\Omega_l = \Re\Omega_l + i\Im\Omega_l \tag{38}$$

Using (38) we can write (37) in the following form:

$$\zeta_k = \sum_{l=1}^{3} (\beta_{kl} e^{i\Re\Omega(\Lambda)t}) e^{-\Im\Omega_l(\Lambda)t}. \tag{39}$$

It may be easily see that for $\Lambda = 0$ we get the solution of the spin motion which is not influenced by the synchrotron radiation. The corresponding $\Omega(0)$-s follow from (32) with $\Lambda = 0$.

We also observe that solution (39) involves term with $\exp\{-\Im\Omega_l(\Lambda)t\}$, where Λ is small parameter. The physical meaning of this term is that it expresses the damping of spin precession caused by the bremsstrahlung. The damping is possible only if $\Im\Omega_l(\Lambda) > 0$. Bayer, Katkov and Fadin used the specific method for determination of such factor for the case of the motion of electron in electromagnetic magnetic field [2, 3]. Baryshevsky [1] applied the Bayer-Katkov-Fadin results for the determination of the polarization of electrons caused by the bent crystals.

The result of the Bayer-Katkov-Fadin method is the term $\exp\{-\delta_l(t/T)\}$, where δ_l are some appropriate constants. They calculated T in the form

$$\frac{1}{T} = \frac{5}{8}\sqrt{3}\alpha\frac{\hbar^2}{m^2}\gamma^5|\dot{\mathbf{v}}|^3, \tag{40}$$

where $\alpha \approx 1/137$ and γ is the Lorentz factor.

We see that we can define T_l by the relations

$$\frac{1}{T_l} = \Im\Omega_l(\Lambda) \tag{41}$$

and for the small parameter Λ it is possible to use approximation

$$\Im\Omega_l(\Lambda) \approx \Im\Omega_l(0) + \Lambda\frac{d\Im\Omega_l(\Lambda)}{d\Lambda}\bigg|_0 + \cdots. \tag{42}$$

In other words, we get also three damping factors as Bayer et al. [3] by the different approach to the bremsstrahlung problem. The method of Schiller and Rafanelli [11] based on the WKB solution of the Dirac equation with bremsstrahlung term was not used by Bayer [2] and Bayer et al. [3]. To our knowledge, the Schiller and Rafanelli method was not still applied to the problem of the influence of the bremsstrahlung on the spin motion.

5 Discussion

We have considered here the influence of the synchrotron radiation on the spin motion of a charged particle moving in the homogeneous magnetic field. It is well known that the synchrotron radiation also influences the trajectory of the charged particle. However we do not consider this influence. It is well known that not only the synchrotron radiation is produced during the motion of a particle in the magnetic field but also the so called spin light, which is generated by spin motion of a particle. We suppose that the influence of the spin light on the spin motion is so small that it is possible to neglect such influence.

The intensity of the synchrotron radiation is, as it is well known, given by the formula [5, 13]:

$$W_{class.\ synch.\ rad.} = \frac{2}{3}\frac{e^2c}{R^2}\gamma^4; \qquad \gamma = \frac{\varepsilon}{m_0c^2}, \tag{43}$$

where R is the radius of the circular motion, ε is the energy of the moving particle.

The intensity of the spin light is expressed by the formula:

$$W_{spin\ light} = \frac{2}{3}\frac{1}{c^3}\left(\frac{d^2}{dt^2}\boldsymbol{\mu}\right)^2 = \frac{2}{3}\frac{\mu_0^2}{c^3}\omega_R^4\zeta_\perp^2. \tag{44}$$

After comparison of formula (28) and (29), we see that the intensity of the spin light is smaller than the intensity of the synchrotron radiation. So, the influence of the spin light on the spin motion can be neglected.

There is the second possibility how to generalize the BMT equation. It consists in axialization of the bremsstrahlung force in the following way:

$$g_\mu(axial) = \frac{2e^3}{3mc^3}\frac{\partial F_{\mu\nu}}{\partial x^\alpha}u^\nu a^\alpha - \frac{2e^4}{3m^2c^5}F_{\mu\alpha}F^{\beta\alpha}a_\beta + \frac{2e^4}{3m^2c^5}(F_{\alpha\beta}u^\beta)(F^{\alpha\gamma}u_\gamma)a_\mu. \tag{45}$$

Then, such force multiplied with the appropriate constant can be add to the original BMT equation. We think that the second conjecture which is presented in this article cannot be a priori excluded.

The verification of the bremsstrahlung equation (16) can be evidently verified by all circular accelerators over the world, including the most gigantic LHC which started its activity by 10.9.2008.

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